

AD-A063 567

BROWN UNIV PROVIDENCE R I LEFSCHETZ CENTER FOR DYNAM--ETC F/G 20/13  
STABILITY OF MOTIONS OF THERMOELASTIC FLUIDS, (U)  
NOV 78 C M DAFERMOS

N00014-76-C-0278

NL

UNCLASSIFIED

| OF |  
ADA  
063567



END  
DATE  
FILMED

3 -79  
DDC

AD A063567

DDC FILE COPY.

STABILITY OF MOTIONS OF THERMOELASTIC FLUIDS

LEVEL

by

C. M. Dafermos

Lefschetz Center for Dynamical Systems  
Division of Applied Mathematics  
Brown University  
Providence, Rhode Island 02912

November, 1978

N00014-76-C-0278  
✓ DAAG27-76-G-0294

DISTRIBUTION STATEMENT A

Approved for public release;  
Distribution Unlimited

DDC

JAN 22 1979

+ This research was supported in part by the National Science Foundation under MCS-76-07247-A03, in part by United States Army under ARO-DAAG27-76-G-0294 and in part by the Office of Naval Research under N00014-76-C-0278-P0002.

78 11 22 019

401 834

LB

# STABILITY OF MOTIONS OF THERMOELASTIC FLUIDS

C. M. Dafermos

↓  
ABSTRACT: It is shown that the second law of thermodynamics induces uniqueness and continuous dependence upon initial state and supply terms of smooth thermodynamic processes of thermoelastic fluids within the broader class of thermodynamic processes with shock waves.  
↖

ACCESSION No.	
RTIN	Whole Section <input checked="" type="checkbox"/>
DD	Half Section <input type="checkbox"/>
UNANNOUNCED	<input type="checkbox"/>
JUSTIFICATION	
<i>Little on file</i>	
BY	
DISTRIBUTION/AVAILABILITY CODES	
Dist.	AVAIL. ORG. OF SPECIAL
A	

A  
78 11 22 019

## 1. Introduction

The intent of this article is to contribute to the project of elucidating the relationship between the second law of thermodynamics and stability. The early work of Duhem [1] and its subsequent developments by Ericksen [2,3], Coleman and Dill [4] and Gurtin [5] indicate that the second law (the Clausius-Duhem inequality) induces Liapunov stability of the equilibrium process in thermomechanics. In a different direction, the author [6] shows that the second law renders uniqueness of thermoelastic processes as well as continuous dependence upon the initial state and the supply terms.

Here we continue the investigation in [6] by examining the implications of the second law on the stability of adiabatic processes of thermoelastic fluids. This problem is of some interest for the following reason: The nonlinearity and hyperbolicity of the field equations, expressing the balance laws of mass, momentum and energy in the theory of thermoelastic fluids, causes the breakdown of smooth solutions and the development of shock waves. Thus, the class of smooth functions is far too narrow to encompass all processes of physical interest. Careful mathematical investigation [7] reveals that processes should be defined in the set of functions of bounded variation, in the sense of Tonelli and Cesari, that is the class of velocity, density and entropy fields whose gradients are Borel measures. The reader who does not care to get involved with technicalities, may, without much loss, visualize these processes as smooth except on a family of propagating surfaces (shocks) across which the velocity, density and entropy fields experience jump discontinuities.



Within the class of processes of bounded variation, the initial value problem admits, in general, many solutions, i.e., there may be many processes that originate from the same state and satisfy the balance laws of mass, momentum and energy for the same body force and heat source. Interpreted as an admissibility criterion, the second law of thermodynamics, in the form of the Clausius-Duhem inequality, rules out some but not necessarily all extraneous processes. Consequently, in order to single out the physically relevant process one has to employ stronger admissibility criteria [8]. In this paper, however, we show that, somewhat surprisingly, whenever a smooth process exists then it is unique and stable within the broader class of processes of bounded variation that satisfy the Clausius-Duhem inequality. In other words, for as long as one is dealing with smooth processes, the second law of thermodynamics in its traditional form is sufficiently powerful to rule out all extraneous processes.

The proof of the aforementioned result is based on an energy inequality, derived in Section 3, in the spirit of the "entropy" estimate established by DiPerna in his interesting paper [9] on uniqueness of solutions of quasilinear hyperbolic systems.

## 2. Thermoelastic Fluids

We begin with a review of the classical theory of thermoelastic fluids. We will be concerned here exclusively with adiabatic processes so that the state variables are velocity  $\underline{v}$ , specific volume  $\tau$  (the inverse of density  $\rho$ , i.e.,  $\tau = 1/\rho$ ), and entropy  $\eta$ . In terms of these, internal energy  $\epsilon$ , pressure  $p$ , and temperature  $\theta$  are determined via constitutive equations:

$$\epsilon = \epsilon^*(\tau, \eta), \quad p = p^*(\tau, \eta), \quad \theta = \theta^*(\tau, \eta) \quad (2.1)$$

$$p^*(\tau, \eta) = - \frac{\partial \epsilon^*(\tau, \eta)}{\partial \tau}, \quad \theta^*(\tau, \eta) = \frac{\partial \epsilon^*(\tau, \eta)}{\partial \eta} \quad (2.2)$$

An admissible thermodynamic process is determined by velocity, specific volume and entropy fields  $(\underline{v}(\underline{x}, t), \tau(\underline{x}, t), \eta(\underline{x}, t))$  that satisfy the balance laws of mass, momentum and energy as well as the second law of thermodynamics, viz.,

$$\frac{\delta \rho}{\delta t} = 0 \quad (2.3)$$

$$\frac{\delta(\rho \underline{v})}{\delta t} + \nabla p = \rho \underline{f} \quad (2.4)$$

$$\frac{\delta}{\delta t} \left[ \frac{1}{2} \rho \underline{v} \cdot \underline{v} + \rho \epsilon \right] + \nabla \cdot (p \underline{v}) = \rho \underline{f} \cdot \underline{v} + \rho r \quad (2.5)$$

$$\frac{\delta(\rho \eta)}{\delta t} - \rho \frac{r}{\theta} \geq 0 \quad (2.6)$$

where  $\underline{f}$  is the body force,  $r$  is the heat source and we are using the notation

$$\frac{\delta \chi}{\delta t} = \chi_t + \nabla \cdot (\chi \underline{v}). \quad (2.7)$$

As explained in the introduction, the fields  $(\underline{v}(\underline{x}, t), \tau(\underline{x}, t), \eta(\underline{x}, t))$  will in general be functions of bounded variation, having shock discontinuities, and, consequently, formulas (2.3)-(2.6) will only be satisfied in a generalized sense (in the sense of measures or distributions). In particular, the classical product differentiation formula does not generally hold for functions in this class so that (2.3)-(2.5) cannot be simplified in the standard fashion. However, in the special case of a Lipschitz continuous thermodynamic process  $(\bar{\underline{v}}(\underline{x}, t), \bar{\tau}(\underline{x}, t), \bar{\eta}(\underline{x}, t))$  one has the reduced form of the balance laws:

$$\bar{\rho}_t + \bar{\underline{v}} \cdot \nabla \bar{\rho} = -\bar{\rho} \nabla \cdot \bar{\underline{v}} \quad (2.8)$$

$$\bar{\underline{v}}_t + \nabla \bar{\underline{v}} \cdot \bar{\underline{v}} + \bar{\tau} \nabla \bar{p} = \bar{\underline{f}} \quad (2.9)$$

$$\bar{\epsilon}_t + \bar{\underline{v}} \cdot \nabla \bar{\epsilon} + \bar{\tau} \bar{p} \nabla \cdot \bar{\underline{v}} = \bar{r}. \quad (2.10)$$

Since  $\bar{\tau} = 1/\bar{\rho}$ , one obtains from (2.8)

$$\bar{\tau}_t + \bar{\underline{v}} \cdot \nabla \bar{\tau} = \bar{\tau} \nabla \cdot \bar{\underline{v}}. \quad (2.11)$$

Furthermore, using (2.2) one easily derives from (2.10)



$$\bar{\eta}_t + \bar{v} \cdot \nabla \bar{\eta} = \frac{\bar{r}}{\bar{\theta}} \quad (2.12)$$

which shows that the Clausius-Duhem inequality (2.6) holds as an equality within this class of processes. We should remark here that Lipschitz continuous processes may have weak (i.e., acceleration) waves but not shock waves.



### 3. Stability of Smooth Solutions

We assume that  $(\bar{v}(x,t), \bar{\tau}(x,t), \bar{\eta}(x,t))$  is a Lipschitz continuous process, with body force  $\bar{f}(x,t)$  and heat source  $\bar{r}(x,t)$ , and let  $(v(x,t), \tau(x,t), \eta(x,t))$  be any other admissible process of bounded variation, with body force  $f(x,t)$  and heat source  $r(x,t)$ . We introduce the functions:

$$H = \rho \left[ \frac{1}{2} (v - \bar{v}) \cdot (v - \bar{v}) + \varepsilon - \bar{\varepsilon} + \bar{p}(\tau - \bar{\tau}) - \bar{\theta}(\eta - \bar{\eta}) \right], \quad (3.1)$$

$$\tilde{F} = H v + (p - \bar{p})(v - \bar{v}). \quad (3.2)$$

We will be using  $H$  as an estimate of the "distance" between the two processes while  $\tilde{F}$  is the "flux" of  $H$ . In order to determine how  $H$  evolves in time, we have to estimate the expression  $H_t + \nabla \cdot \tilde{F}$ . On account of (2.7),

$$H_t + \nabla \cdot \tilde{F} = \frac{\delta H}{\delta t} + \nabla \cdot [(p - \bar{p})(v - \bar{v})]. \quad (3.3)$$

Using the identity

$$\frac{\delta(\chi\psi)}{\delta t} = \psi \frac{\delta\chi}{\delta t} + \chi(\psi_t + v \cdot \nabla \psi), \quad (3.4)$$

which holds in the sense of measures whenever  $\chi$  is any function of bounded variation and  $\psi$  is Lipschitz continuous, one obtains

$$\begin{aligned}
H_t + \nabla \cdot \tilde{F} &= \frac{\delta}{\delta t} \left[ \frac{1}{2} \rho \underline{v} \cdot \underline{v} + \rho \varepsilon \right] - \underline{\bar{v}} \cdot \frac{\delta(\rho \underline{v})}{\delta t} - \rho \underline{v} \cdot [\underline{\bar{v}}_t + \nabla \underline{\bar{v}} \cdot \underline{\bar{v}}] \\
&+ \rho \underline{\bar{v}} \cdot [\underline{\bar{v}}_t + \nabla \underline{\bar{v}} \cdot \underline{\bar{v}}] - \rho [\underline{\bar{\varepsilon}}_t + \underline{v} \cdot \nabla \underline{\bar{\varepsilon}}] + \underline{\bar{p}}_t + \nabla \cdot (\underline{\bar{p}} \underline{v}) \\
&- \rho \underline{\bar{p}} [\underline{\bar{\tau}}_t + \underline{v} \cdot \nabla \underline{\bar{\tau}}] - \rho \underline{\bar{\tau}} [\underline{\bar{p}}_t + \underline{v} \cdot \nabla \underline{\bar{p}}] - \underline{\bar{\theta}} \frac{\delta(\rho \eta)}{\delta t} \\
&- \rho \eta [\underline{\bar{\theta}}_t + \underline{v} \cdot \nabla \underline{\bar{\theta}}] + \rho \underline{\bar{\eta}} [\underline{\bar{\theta}}_t + \underline{v} \cdot \nabla \underline{\bar{\theta}}] + \rho \underline{\bar{\theta}} [\underline{\bar{\eta}}_t + \underline{v} \cdot \nabla \underline{\bar{\eta}}] \\
&+ \nabla \cdot (\underline{p} \underline{v}) - \nabla \cdot (\underline{p} \underline{\bar{v}}) - \nabla \cdot (\underline{\bar{p}} \underline{v}) + \nabla \cdot (\underline{\bar{p}} \underline{\bar{v}}).
\end{aligned} \tag{3.5}$$

With the help of the balance laws (2.3)-(2.6) and (2.9) we get

$$\begin{aligned}
H_t + \nabla \cdot \tilde{F} &\leq -\nabla \cdot (\underline{p} \underline{v}) + \rho \underline{f} \cdot \underline{v} + \rho r - \underline{\bar{v}} \cdot [-\nabla \underline{p} + \rho \underline{f}] \\
&- \rho \underline{v} \cdot [-\underline{\bar{\tau}} \nabla \underline{\bar{p}} + \underline{\bar{f}}] + \rho \underline{\bar{v}} \cdot [-\underline{\bar{\tau}} \nabla \underline{\bar{p}} + \underline{\bar{f}}] \\
&- \rho (\underline{v} - \underline{\bar{v}}) \cdot \nabla \underline{\bar{v}} \cdot (\underline{v} - \underline{\bar{v}}) + \underline{\bar{p}}_t + \nabla \cdot (\underline{\bar{p}} \underline{v}) - \rho \underline{\bar{\tau}} [\underline{\bar{p}}_t + \underline{v} \cdot \nabla \underline{\bar{p}}] \\
&- \rho \{ \underline{\bar{\varepsilon}}_t + \underline{v} \cdot \nabla \underline{\bar{\varepsilon}} + \underline{\bar{p}} [\underline{\bar{\tau}}_t + \underline{v} \cdot \nabla \underline{\bar{\tau}}] - \underline{\bar{\eta}} [\underline{\bar{\theta}}_t + \underline{v} \cdot \nabla \underline{\bar{\theta}}] \} \\
&- \rho r \frac{\bar{\theta}}{\bar{\theta}} - \rho (\eta - \underline{\bar{\eta}}) [\underline{\bar{\theta}}_t + \underline{v} \cdot \nabla \underline{\bar{\theta}}] \\
&+ \nabla \cdot (\underline{p} \underline{v}) - \nabla \cdot (\underline{p} \underline{\bar{v}}) - \nabla \cdot (\underline{\bar{p}} \underline{v}) + \nabla \cdot (\underline{\bar{p}} \underline{\bar{v}}).
\end{aligned} \tag{3.6}$$

We simplify the right-hand side of (3.6), noting, in particular, that, by virtue of (2.2),

$$\bar{\epsilon}_t + \bar{p}\bar{\tau}_t - \bar{\eta}\bar{\theta}_t = 0 \quad (3.7)$$

$$\nabla \bar{\epsilon} + \bar{p}\nabla \bar{\tau} - \bar{\eta}\nabla \bar{\theta} = 0, \quad (3.8)$$

thus arriving at

$$H_t + \nabla \cdot \bar{F} \leq \rho(\bar{f} - \bar{f}) \cdot (\bar{v} - \bar{v}) - \rho(\bar{v} - \bar{v}) \cdot \nabla \bar{v} \cdot (\bar{v} - \bar{v}) \quad (3.9)$$

$$+ \rho r - \rho r \frac{\bar{\theta}}{\theta} + (1 - \rho\bar{\tau}) [\bar{p}_t + \bar{v} \cdot \nabla \bar{p}] + (\bar{p} - p) \nabla \cdot \bar{v}$$

$$- \rho(\eta - \bar{\eta}) [\bar{\theta}_t + \bar{v} \cdot \nabla \bar{\theta}] - \rho(\eta - \bar{\eta}) (\bar{v} - \bar{v}) \cdot \nabla \bar{\theta}$$

$$= \rho(\bar{f} - \bar{f}) \cdot (\bar{v} - \bar{v}) - \rho(\bar{v} - \bar{v}) \cdot \nabla \bar{v} \cdot (\bar{v} - \bar{v})$$

$$+ \rho r - \rho r \frac{\bar{\theta}}{\theta} + \rho(\tau - \bar{\tau}) \left\{ \frac{\partial p^*}{\partial \tau} [\bar{\tau}_t + \bar{v} \cdot \nabla \bar{\tau}] + \frac{\partial p^*}{\partial \eta} [\bar{\eta}_t + \bar{v} \cdot \nabla \bar{\eta}] \right\}$$

$$+ (\bar{p} - p) \nabla \cdot \bar{v} - \rho(\eta - \bar{\eta}) \left\{ \frac{\partial \theta^*}{\partial \tau} [\bar{\tau}_t + \bar{v} \cdot \nabla \bar{\tau}] + \frac{\partial \theta^*}{\partial \eta} [\bar{\eta}_t + \bar{v} \cdot \nabla \bar{\eta}] \right\}$$

$$- \rho(\eta - \bar{\eta}) (\bar{v} - \bar{v}) \cdot \nabla \bar{\theta}.$$

On account of (2.2),

$$\frac{\partial p^*}{\partial \eta} = - \frac{\partial \theta^*}{\partial \tau} = \frac{\partial^2 \epsilon^*}{\partial \tau \partial \eta} \quad (3.10)$$

so that, with the help of (2.11) and (2.12), we may rewrite (3.9) in the form

$$\begin{aligned}
H_t + \nabla \cdot \underline{F} \leq & \rho(\underline{f} - \bar{f}) \cdot (\underline{v} - \bar{v}) - \rho(\underline{v} - \bar{v}) \cdot \nabla \bar{v} \cdot (\underline{v} - \bar{v}) \\
& - [p - \bar{p} - \frac{\partial p^*}{\partial \tau} (\tau - \bar{\tau}) - \frac{\partial p^*}{\partial \eta} (\eta - \bar{\eta})] \nabla \cdot \bar{v} \\
& - \rho \frac{\partial p^*}{\partial \tau} (\tau - \bar{\tau})^2 \nabla \cdot \bar{v} - \rho \frac{\partial p^*}{\partial \eta} (\tau - \bar{\tau}) (\eta - \bar{\eta}) \nabla \cdot \bar{v} \\
& + \rho \frac{\bar{r}}{\bar{\theta}} [\theta - \bar{\theta} - \frac{\partial \theta^*}{\partial \tau} (\tau - \bar{\tau}) - \frac{\partial \theta^*}{\partial \eta} (\eta - \bar{\eta})] \\
& - \rho \frac{\bar{r}}{\bar{\theta}} \frac{1}{\bar{\theta}} (\theta - \bar{\theta})^2 + \rho \frac{\bar{r} - \bar{r}}{\bar{\theta}} (\theta - \bar{\theta}) - \rho(\eta - \bar{\eta}) (\underline{v} - \bar{v}) \cdot \nabla \bar{\theta}.
\end{aligned} \tag{3.11}$$

Estimate (3.11) will be the tool for establishing stability. The crucial observation is that, by virtue of (2.2),  $H$  and  $\underline{F}$  are of quadratic order in the differences  $(\underline{v} - \bar{v}, \tau - \bar{\tau}, \eta - \bar{\eta})$  and that the right-hand side of (3.11) is of quadratic order in  $(\underline{v} - \bar{v}, \tau - \bar{\tau}, \eta - \bar{\eta}, \underline{f} - \bar{f}, r - \bar{r})$ . In addition to that we have to make certain that  $H$  is positive definite. To this end, we impose on the constitutive equations the following restrictions:

$$\left(\frac{\partial p}{\partial \tau}\right)_\eta < 0, \quad \left(\frac{\partial \theta}{\partial \eta}\right)_p > 0. \tag{3.12}$$

In (3.12) we are using the notation of classical thermodynamics, visualizing  $p$  as a function of  $(\tau, \eta)$  and taking the partial derivative with respect to  $\tau$  and then visualizing  $\theta$  as a function of  $(p, \eta)$  and taking the partial derivative with respect to  $\eta$ . Assumption (3.12) is physically reasonable, for as long as the fluid does not undergo any phase transitions, and its connection with Gibb's stability has been established in classical thermodynamics. In particular, (3.12) is satisfied by the constitutive equations of polytropic gases.



In order to see the implications of (3.12) on the function  $\varepsilon^*(\tau, \eta)$ , note first that, by virtue of (2.2),

$$\left(\frac{\partial p}{\partial \tau}\right)_\eta = - \frac{\partial^2 \varepsilon^*}{\partial \tau^2}. \quad (3.13)$$

Furthermore, by chain rule,

$$\begin{aligned} \left(\frac{\partial \theta}{\partial \eta}\right)_p &= \left(\frac{\partial \theta}{\partial \eta}\right)_\tau - \left(\frac{\partial \theta}{\partial p}\right)_\eta \left(\frac{\partial p}{\partial \eta}\right)_\tau \\ &= \left(\frac{\partial \theta}{\partial \eta}\right)_\tau - \left(\frac{\partial \theta}{\partial \tau}\right)_\eta \left(\frac{\partial p}{\partial \tau}\right)_\eta^{-1} \left(\frac{\partial p}{\partial \eta}\right)_\tau \\ &= \left(\frac{\partial p}{\partial \tau}\right)_\eta^{-1} \left\{ \left(\frac{\partial p}{\partial \tau}\right)_\eta \left(\frac{\partial \theta}{\partial \eta}\right)_\tau - \left(\frac{\partial \theta}{\partial \tau}\right)_\eta \left(\frac{\partial p}{\partial \eta}\right)_\tau \right\} \\ &= \left(\frac{\partial^2 \varepsilon^*}{\partial \tau^2}\right)^{-1} \left\{ \frac{\partial^2 \varepsilon^*}{\partial \tau^2} \frac{\partial^2 \varepsilon^*}{\partial \eta^2} - \left(\frac{\partial^2 \varepsilon^*}{\partial \tau \partial \eta}\right)^2 \right\}. \end{aligned} \quad (3.14)$$

It is clear from (3.13) and (3.14) that (3.12) holds if and only if the function  $\varepsilon^*(\tau, \eta)$  is (locally) uniformly convex.

Let us now assume that the fluid is confined in a vessel with rigid boundary that occupies a smooth bounded domain  $\Omega \subset \mathbb{R}^n$  ( $n = 1, 2$  or  $3$ ). This leads to boundary conditions

$$\underline{v} \cdot \underline{v} = 0 \quad (3.15)$$

where  $\underline{v}$  is the unit normal on the boundary of  $\Omega$ . Our stability result is given in the following

**Theorem.** Let  $(\bar{v}(\underline{x}, t), \bar{\tau}(\underline{x}, t), \bar{\eta}(\underline{x}, t))$  be a bounded and uniformly Lipschitz continuous process defined for  $\underline{x} \in \bar{\Omega}$ ,  $t \geq 0$  and satisfying the balance laws (2.8)-(2.10) and the boundary condition

$\bar{v} \cdot \underline{v} = 0$ . Consider any admissible process  $(\underline{v}(\underline{x}, t), \tau(\underline{x}, t), \eta(\underline{x}, t))$  of bounded variation defined for  $\underline{x} \in \bar{\Omega}$ ,  $t \geq 0$  and satisfying the balance laws (2.3)-(2.6), the boundary condition  $\underline{v} \cdot \underline{v} = 0$  and the bounds

$$|\tau(\underline{x}, t)|, |\rho(\underline{x}, t)|, |\eta(\underline{x}, t)|, |\theta^{-1}(\underline{x}, t)| \leq C, \quad \underline{x} \in \bar{\Omega}, t \geq 0. \quad (3.16)$$

Then there are positive constants  $A$  and  $\alpha$ , depending solely upon bounds of  $(\bar{v}, \bar{\tau}, \bar{\eta})$  and its Lipschitz constant, the function  $\varepsilon^*(\tau, \eta)$  and the constant  $C$  in (3.16), such that, for any  $t \geq 0$ ,

$$\begin{aligned} & \int_{\Omega} \{ |\underline{v}(\underline{x}, t) - \bar{v}(\underline{x}, t)|^2 + |\tau(\underline{x}, t) - \bar{\tau}(\underline{x}, t)|^2 + |\eta(\underline{x}, t) - \bar{\eta}(\underline{x}, t)|^2 \} d\underline{x} \\ & \leq Ae^{\alpha t} \left[ \int_{\Omega} \{ |\underline{v}(\underline{x}, 0) - \bar{v}(\underline{x}, 0)|^2 + |\tau(\underline{x}, 0) - \bar{\tau}(\underline{x}, 0)|^2 + |\eta(\underline{x}, 0) - \bar{\eta}(\underline{x}, 0)|^2 \} d\underline{x} \right. \\ & \quad \left. + \int_0^t \int_{\Omega} \{ |\underline{f}(\underline{x}, s) - \bar{f}(\underline{x}, s)|^2 + |r(\underline{x}, s) - \bar{r}(\underline{x}, s)|^2 \} d\underline{x} ds \right]. \end{aligned} \quad (3.17)$$

Proof. Since the right-hand side of (3.11) is of quadratic order in  $(\underline{v} - \bar{v}, \tau - \bar{\tau}, \eta - \bar{\eta}, \underline{f} - \bar{f}, r - \bar{r})$ , we have

$$H_t + \nabla \cdot \underline{F} \leq b \{ |\underline{v} - \bar{v}|^2 + |\tau - \bar{\tau}|^2 + |\eta - \bar{\eta}|^2 + |\underline{f} - \bar{f}|^2 + |r - \bar{r}|^2 \} \quad (3.18)$$

with  $b$  depending solely upon bounds of  $(\bar{v}, \bar{\tau}, \bar{\eta})$  and its Lipschitz constant, the function  $\varepsilon^*(\tau, \eta)$  and  $C$  in (3.16). Integrating (3.18) over  $\Omega \times [0, t]$ , applying Green's theorem and noting that by (3.2) and the boundary conditions we have  $\underline{F} \cdot \underline{v} = 0$ , we obtain

$$\int_{\Omega} H(\underline{x}, t) d\underline{x} - \int_{\Omega} H(\underline{x}, 0) d\underline{x} \quad (3.19)$$

$$\leq b \int_0^t \int_{\Omega} \{ |\underline{v} - \bar{\underline{v}}|^2 + |\tau - \bar{\tau}|^2 + |\eta - \bar{\eta}|^2 + |\underline{f} - \bar{\underline{f}}|^2 + |\underline{r} - \bar{\underline{r}}|^2 \} d\underline{x} ds.$$

Now  $H$  is of quadratic order in  $(\underline{v} - \bar{\underline{v}}, \tau - \bar{\tau}, \eta - \bar{\eta})$  so that

$$\begin{aligned} \int_{\Omega} H(\underline{x}, 0) d\underline{x} &\leq M \int_{\Omega} \{ |\underline{v}(\underline{x}, 0) - \bar{\underline{v}}(\underline{x}, 0)|^2 \\ &\quad + |\tau(\underline{x}, 0) - \bar{\tau}(\underline{x}, 0)|^2 + |\eta(\underline{x}, 0) - \bar{\eta}(\underline{x}, 0)|^2 \} d\underline{x}. \end{aligned} \quad (3.20)$$

On the other hand, since  $\varepsilon^*(\tau, \eta)$  is uniformly convex,  $H$  is positive definite and so

$$\begin{aligned} \int_{\Omega} H(\underline{x}, t) d\underline{x} &\geq m \int_{\Omega} \{ |\underline{v}(\underline{x}, t) - \bar{\underline{v}}(\underline{x}, t)|^2 \\ &\quad + |\tau(\underline{x}, t) - \bar{\tau}(\underline{x}, t)|^2 + |\eta(\underline{x}, t) - \bar{\eta}(\underline{x}, t)|^2 \} d\underline{x}. \end{aligned} \quad (3.21)$$

The positive constants  $M$  and  $m$  in (3.20) and (3.21) depend on the same factors as  $b$  in (3.17). We now combine (3.19), (3.20), (3.21) and we apply Gronwall's inequality (e.g. [10, I.6.6]) thus arriving at (3.17) with  $\alpha = b/m$  and  $A = \max\{M/m, b/m\}$ . This completes the proof.

The preceeding proposition yields immediately the following result on uniqueness:

Corollary. Let  $(\bar{v}(x,t), \bar{\tau}(x,t), \bar{\eta}(x,t))$  be a uniformly Lipschitz continuous process defined for  $x \in \bar{\Omega}$ ,  $t \geq 0$  and satisfying the balance laws (2.8)-(2.10) and the boundary condition  $\bar{v} \cdot \nu = 0$ . Then there is no other admissible process (even in the broader class of functions of bounded variation) that satisfies the same initial and boundary conditions and has the same body force and heat source as the process  $(\bar{v}(x,t), \bar{\tau}(x,t), \bar{\eta}(x,t))$ .

One may also establish uniqueness and stability results for the case where the fluid occupies the entire space by applying the techniques developed in [6].



## REFERENCES

- [1] P. Duhem, Traité d'Énergetique ou Thermodynamique Générale, Vol. 2, Paris (1911).
- [2] J. L. Ericksen, A thermo-kinetic view of elastic stability theory, Int. J. Solids Structures 2, 573-580(1966).
- [3] J. L. Ericksen, Thermoelastic stability, Proc. 5th U.S. National Cong. Appl. Mech., 187-193(1966).
- [4] B. D. Coleman and E. H. Dill, On thermodynamics and the stability of motions of materials with memory, Arch. Rational Mech. Analysis 51, 1-13(1973).
- [5] M. E. Gurtin, Thermodynamics and stability, Arch. Rational Mech. Analysis, 59, 63-96(1975).
- [6] C. M. Dafermos, The second law of thermodynamics and stability (to appear).
- [7] A. I. Volpert, The space BV and quasilinear equations, Mat. Sbornik (N.S.) 73(115), 255-302(1967). English translation: Math. USSR-Sbornik 2, 225-267(1967).
- [8] C. M. Dafermos, The entropy rate admissibility criterion in thermoelasticity, Rend. Accad. Naz. Lincei, Ser. VIII, 57, 113-119(1974).
- [9] R. J. DiPerna, Uniqueness of solutions to hyperbolic conservation laws (to appear).
- [10] J. K. Hale, Ordinary Differential Equations, Wiley, New York (1968).